For the stiffnesses in the axial mode, the exact solution for the sphere-in-sphere[1] can be shown to be

$$K = \frac{W}{w_0} = \frac{8\pi cG}{3(1-R)} f(\nu,R) \tag{29}$$

where

$$f(\nu,R) = \frac{3(1-\nu)(5-6\nu)}{9 - 20\nu + 12\nu^2 - 2(1-R^3)^2/3(1-R)(1-R^5)}$$

The resulting stiffness can be compared with the corresponding approximate solution [Eq. (27)]. These results differ in the last term only. Consequently, these factors alone are used in the comparative study (results listed in Table 1). The exact solution portion, $f(\nu,R)$, is a function of R while its approximate counterpart is not. The approximate solution agrees within 6% for $R \geqq \frac{1}{2}$ which represents the range of practical interest ($R = 0$ implies an infinitely large filler). The agreement is closer using smaller values of Poisson's ratio which happen to comply well with the assumption made in the approximate closed-form solution.

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An Evaluation of a Transfer Coefficient Approach for Unequal-Diffusion Coefficients

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1. Introduction

IN a recent AIAA paper,[1] Kendall, Rindal, and Bartlett postulated a procedure that provides a significant simplification for solving ablative-material-coupled, unequal-diffusion, multicomponent boundary-layer problems. The procedure employs correlation equations that directly relate boundary-layer edge conditions to the ablative surface condition through transfer coefficients and driving potentials. The form of the driving potentials was suggested by terms in the differential equations that were derived using a special treatment of diffusion coefficients (termed a bifurcation approximation).

It is the purpose of this Note to describe the extent to which successful validation of this approach has been achieved and to present values for the arbitrary parameters involved. This was accomplished using the results of "exact" solutions of the differential equations obtained with the numerical solution procedure described in Ref. 2. Solutions were obtained for two representative materials, graphite and nylon phenolic, and five flight conditions at the stagnation point of a hemispherical re-entry body having a radius of 1 in. In these calculations, the boundary layer is fully coupled with the ablative-material response; the wall boundary condition satisfying the elemental mass balances, a steady-state energy balance, and surface equilibrium.

2. Summary of the Unequal-Diffusion Film Analogy

The extension of the simple film transport concept to describe unequal-diffusion effects was based on a modeling of the governing differential equations describing the process. Such modeling was accomplished in Ref. 1 to which the reader seeking a description of the mathematical development is directed. The pertinent concepts of the method are:

1) In the bifurcation approximation to the boundary-layer elemental species equation, the diffusive term contains the gradient of a newly-defined species elemental fraction, $\tilde{Z}_k$, (the elemental mass fraction still appearing in the convection terms). The analogy between this equation and its equal-diffusion counterpart, in conjunction with a film-transport correlation equation for equal diffusion, leads to the following correlation equation for unequal diffusion:

$$\tilde{j}_{k_w} = \rho_e U_e C_{M_k}(\tilde{Z}_{k_w}{}^* - \tilde{Z}_{k_e}{}^*) \tag{1}$$

The driving potential $\tilde{Z}_k{}^*$ is a geometrically weighted average between the driving potentials applicable to the convective and diffusional terms that appear in the differential equation. An arbitrary weighting factor γ suggested in Ref. 1 to have a value of $\frac{2}{3}$, appears in the definition for $\tilde{Z}_k{}^*$. A primary purpose for introducing the $\tilde{Z}_k{}^*$ potential is to attempt to reduce the C_{M_k} to a single C_M through appropriate choice of γ. The success of this approach is one of the principal findings of this Note.

2) The wall energy flux is conveniently divided into terms representing two processes: diffusion of chemical energy at some base temperature and convection of sensible enthalpy measured above this base temperature.[3] In Ref. 1, a film-coefficient correlation equation is suggested having terms corresponding to these two processes. The first is represented by a mean (or over-all effective) mass-transfer coefficient and a weighted species-enthalpy driving potential, the second by a heat-transfer coefficient and a sensible enthalpy driving potential. The form of the equation presented in Ref. 1 is based on the arbitrary choice of the wall temperature as the base temperature. More generally, the energy equation with arbitrary base temperature is

$$q_w = \rho_e U_e C_M \sum_i (Z_{ie}{}^* - Z_{iw}{}^*)h_{ib} + \rho_e U_e C_H \times \left[(H_{s_r} - h_{sb})_e - \sum_i K_{iw}(h_{iw} - h_{ib})\right] \tag{2}$$

where K_i and h_i are the mass fraction and enthalpy of species i, h_s and H_{s_r} are sensible and recovery enthalpies, and the b, w, and e subscripts denote base temperature, wall, and boundary-layer edge composition, respectively. The influence of the base temperature also is established in this Note.

3. Mass-Transfer Coefficient Results

Two ablative materials, graphite and nylon phenolic, were evaluated over a range of representative earth-ballistic-reentry conditions using computer codes in which both the exact and foregoing film-analogy equations were programed.[2,4] A summary of the elemental mass-transfer coefficient results are presented in Fig. 1. These results were obtained from Eq. (1) with both the elemental flux at the wall $\tilde{j}_{k_w}$ and the driving potentials $\tilde{Z}_k{}^*$ obtained from the "exact" steady-state solutions.

For the three-element graphite solutions, precise crossovers occur since there are only N-1 = 2 independent elemental conservation equations.[5] The crossover points are seen to occur at values of γ varying from 0.3–0.7. In the case of the four-element nylon-phenolic solutions, unique crossovers are not generally obtained; however, the curves do demonstrate

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a minimum spread region for γ values of 0.5–0.8. The variation in C_{M_k} about the arithmetic mean is seen to be less than 20% in this region. The effect of these results on the accuracy of ablation-response predictions is evaluated in Sec. 5.

4. Heat-Transfer Coefficient Results

Since a unique value for the mass-transfer coefficient is desirable and, in fact, demanded by the energy-flux correlation equation (2), some method for averaging the C_{M_k} values produced by Eq. (1) must be used. Three techniques for averaging the C_{M_k} were studied: arithmetic, mass weighting, and mole weighting. The latter two averages are weighted on the basis of the absolute values of $\tilde{Z}_k{}^*$ and $\tilde{Z}_k{}^*\mathfrak{M}/\mathfrak{M}_k$ driving potentials, respectively, where $\mathfrak{M}$ is the molecular weight of the mixture and $\mathfrak{M}_k$ is the atomic weight of element k. For each of these averages, C_H was calculated from Eq. (2) for three definitions of the base temperature T_b using the wall diffusive heat flux q_w and driving potentials from the "exact" solution. The three temperatures chosen were the wall temperature, the edge temperature, and the average of these two temperatures. The influence of T_b on the resultant value of C_H was found generally to be less than 10% for the conditions studied as shown by the typical results presented in Table 1. Although this is not a dominant effect, it is sufficiently large that it should not be categorically dismissed when performing correlations of boundary-layer-solution data. The influence of the C_{M_k} averaging technique on ablation response is presented in the next section.

5. Ablation Response Comparison

Comparisons were made between the exact and film analogy for steady-state ablation rates $\dot{m}_w$ and surface temperatures T_w. The effects of γ and the C_M averaging techniques were explored for the two materials, with the base temperature set equal to the wall temperature. In the case of graphite, selection of the value of γ for a given flight condition to achieve crossover (see Fig. 1) yields precise prediction of the theoretical ablation response since the $\tilde{j}_{k_w}$ and q_w can be duplicated by Eqs. (1) and (2) without ambiguity regarding $\rho_e U_e C_M$. The accuracy of the film-analogy results for noncrossover values of γ (e.g., $\gamma = \frac{2}{3}$) was also very good for graphite because unequal diffusion effects are relatively small for this material.[5,6]

The percent deviation of the film analogy results from the exact solutions for nylon phenolic are presented in Fig. 2. It is seen that the deviation is least sensitive to the value of γ for the mass-weighted average of the C_{M_k} and that good agreement with the exact solutions can be obtained (within $\pm 2\%$) for any value of γ between 0.5 and 0.8. This is to be compared with about a 10% error on the average assuming equal diffusion.[5] Thus, the value of $\gamma = \frac{2}{3}$ suggested in Ref. 1, though not precise, indeed is appropriate for the flight conditions and ablation materials studied in this Note.

6. Closure

The fundamental validity of the unequal-diffusion film-coefficient formulation proposed in Ref. 1 has been demonstrated by comparing steady-state $\dot{m}_w$ and T_w computed with this approach with that resulting from numerical solution of the boundary-layer equations. The film-coefficient formulation is seen to be most effective when C_M is computed by mass averaging of the C_{M_k}, the C_{M_k} being computed from the boundary-layer solution data. A value of γ equal to $\frac{2}{3}$ and the base temperature equal to the wall temperature, both recommended in Ref. 1, are seen to be appropriate for the ablation materials considered in this Note. Practical use of

Table 1 Values of C_M/C_H for nylon phenolic obtained from exact solution of the boundary-layer equations ($\gamma = 0.7$)

Flight conditions		C_M/C_H (arithmetic mean) for		
Pressure, atm	Enthalpy, Btu/lb	$T_b = T_w$	$T_b = (T_w + T_e)/2$	$T_b = T_e$
1	13,000	0.3733	0.4014	0.4207
100	10,000	0.3541	0.4072	0.4460
1	10,000	0.4021	0.4272	0.4452
0.01	10,000	0.5932	0.5834	0.5720
1	6,000	0.4597	0.4758	0.4881

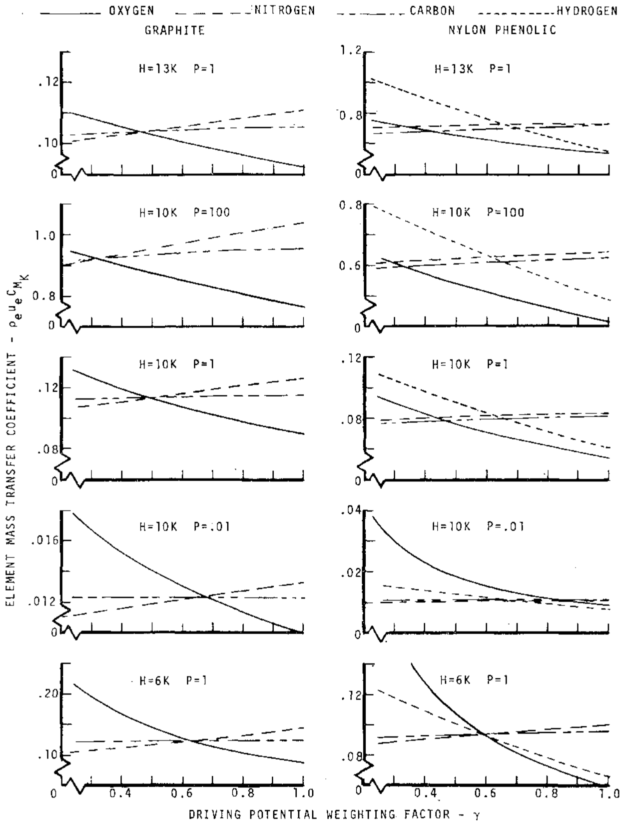

Fig. 1 Effect of driving potential weighting factor γ on elemental mass-transfer coefficients.

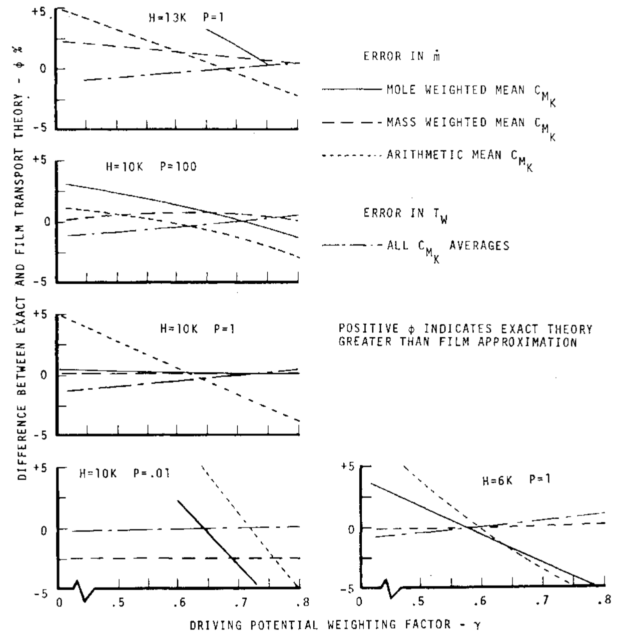

Fig. 2 Effect of mass-transfer-coefficient averaging techniques on predicted ablation rates and surface temperatures.

the approach, however, necessitates the development of engineering correlations for C_M and C_H. Such correlations have been developed for graphite ablation in air and are presented in Ref. 6. Correlations for additional materials are in progress. Results to date suggest that C_M approaches C_H when the system Lewis number (as defined in Ref. 1) approaches unity, but that the conventional relation $[C_M/C_H = (\mathrm{Le})^{2/3}]$ can be in substantial error for large mass-transfer rates, especially when the injectant reacts chemically with the edge gases.[5] Such correlations will eventually provide a means for rapidly and accurately predicting unequal-diffusion ablation response from relatively well-known nonablating heat-transfer coefficients.

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Velocity Laws for Turbulent Boundary Layers with Mass Addition and Combustion

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This Note presents the results of a study directed toward the use of the turbulent kinetic energy equation as the basic closure equation for the problem of a subsonic variable density turbulent boundary layer. The case considered is that of a uniform two-dimensional, turbulent boundary layer with a dilute mixture of hydrogen and nitrogen injected into the boundary layer at the surface and combustion occurring at a flame front in the boundary layer. The method of analysis presented is an extension of the methods employed by Townsend,[1] Bradshaw, et al.,[2] Bradshaw,[3] and Bradshaw and Ferriss.[4]

The turbulent kinetic energy equation is utilized to obtain expressions for the velocity distribution through the boundary layer for the case of a zero axial pressure gradient. Similarity between the concentration, enthalpy, and velocity profiles is employed in order to relate the local density to the local velocity. Experimental results are correlated on the basis of the resulting expressions over the inner region of the turbulent boundary layer. These correlations indicate the energy dissipation length is directly proportional to distance from the surface over approximately 10% of the boundary-layer thickness at the higher blowing rates.

The turbulent kinetic energy equation for a two-dimensional compressible flow may be written as [1,2]

$$\langle\rho\rangle\langle U\rangle\partial\langle q^2/2\rangle/\partial x + \langle\rho\rangle\langle V\rangle\partial\langle q^2/2\rangle/\partial y + \langle\rho uv\rangle\,\partial\langle U\rangle/\partial y + \partial(\langle pv\rangle + \langle\rho q^2v/2\rangle)/\partial y + \langle\rho\rangle\epsilon = 0 \quad (1)$$

where $\langle U\rangle$ and $\langle V\rangle$ are the mean velocities in the x and y directions, respectively, $\langle q^2\rangle = \langle u^2\rangle + \langle v^2\rangle + \langle w^2\rangle$, p is the pressure fluctuation, $\langle\rho\rangle$ is the mean density, u, v, and w are the velocity fluctuations and ϵ is the dissipation of turbulent kinetic energy due to viscous effects. Equation (1) represents the rate of change of $\langle q^2/2\rangle$ along a streamline as the net sum of advection, production, diffusion, and viscous dissipation of the turbulent kinetic energy.

The turbulent kinetic energy equation may be converted into a shear stress equation by defining[1,2]

$$\tau/\langle\rho\rangle = -\langle\rho uv\rangle/\langle\rho\rangle \quad (2a)$$

$$a_1 = \tau/(\langle\rho\rangle\langle q^2\rangle) \quad (2b)$$

$$L = (\tau/\langle\rho\rangle)^{3/2}/\epsilon \quad (2c)$$

and

$$(1/\langle\rho\rangle)\partial(\langle pv\rangle + \langle\rho q^2v/2\rangle)/\partial y = a_2\partial(\langle q^2\rangle)^{3/2}/\partial y \quad (2d)$$

In Eqs. (2) τ is the local shear stress, L is an energy dissipation length, and a_1 and a_2 are constants. The hypothesis of turbulence structural equilibrium as employed by Townsend[1] is used to justify the form of the diffusion term chosen in Eq. (2d).

Thus, for the case of surface mass injection with combustion, we may write, neglecting the variation of $\langle q^2/2\rangle$ in the x direction and setting $V = V_w$,

$$[1/(\tau/\rho)^{1/2}]dU - 1/(2\tau/\rho)d(\tau/\rho) - (V_w/2a_1)[1/(\tau/\rho)^{3/2}]d(\tau/\rho) = (1/L)dy \quad (3)$$

where all brackets denoting average quantities have been dropped.

In obtaining this expression, the factor $3a_2/2(a_1)^{3/2}$ has been set equal to $\frac{1}{2}$, following the arguments of Townsend.[1] It should also be noted that the approximation of $V = V_w$ was found to be necessary in order to achieve the correlations presented. When V is set equal to $(\rho_w/\rho)V_w$, the advection term becomes dominant due to the significant decrease in the density ρ across the inner region of the boundary layer and correlation of the velocity profile with distance from the surface is not achieved. The successful correlation achieved with setting $V = V_w$ may possibly be explained by noting that very near the surface, the effect on the density of the increasing temperature is offset somewhat by the effect of the decreasing concentration of hydrogen and the approximation $V = V_w$ appears to be sufficiently accurate. With V set equal to V_w, the contribution of the advection term is suppressed in the region away from the immediate vicinity of the wall and the production and dissipation terms in the kinetic energy equation become the dominant terms.

For the turbulent boundary layer with mass addition and zero pressure gradient, the mean momentum equation may be written as

$$\tau/\rho = (\rho_w/\rho)(\tau_w/\rho_w + V_wU) \quad (4)$$

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